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COMMENT

Reply to 'Relativistic formulation of quantum state diffusion?'

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Abstract. It has been demonstrated recently by the authors that the non-relativistic quantum state diffusion model can be generalized to yield a consistent, relativistically covariant theory. It is shown here that the 'counterexample' constructed by Diósi can easily be disproven.

1. Introduction

Recently, we have proposed [1–3] a relativistic generalization of the non-relativistic quantum state diffusion model developed by Gisin and Percival [4]. In a recent comment Diósi [5] made an attempt to show, with the help of a 'counterexample', an inconsistency in our theory. We shall demonstrate here that, when applying our theory correctly, Diósi's 'counterexample' can easily be disproven. In fact, as we shall see below Diósi's example involves an erroneous notion of covariance, and it uses incorrect transformation laws for physical observables. Moreover, Diósi obviously missed the most important concept of our theory, namely its Hilbert bundle structure, by which probabilities and expectation values are determined in our theory.

In section 2 we shall briefly outline the most important general facts that led to our formulation of a relativistic quantum state diffusion model for the Dirac electron. These considerations serve to clarify the motivation and the physical content of our theory. Section 3 is devoted to a detailed discussion of Diósi's example. This discussion is performed fully relativistically and it is demonstrated that there are no inconsistencies in our theory.

2. General considerations

In a series of papers [6–8] Aharonov and Albert have demonstrated, with the help of many examples, the following fact. If one takes into account the changes of the state vector induced by local or non-local measurement processes in relativistic quantum theory, the wavefunction ceases to be a function $\psi = \psi(x)$ on the spacetime continuum, but becomes a functional $\psi = \psi(\sigma)$ on the set of spacelike hypersurfaces σ in Minkowski space. This is a far-reaching conclusion which served as a starting point of our approach.

We briefly recall the simplest of the examples which led Aharanov and Albert to this conclusion [8]. Suppose that in the infinite past an electron has been prepared in a state

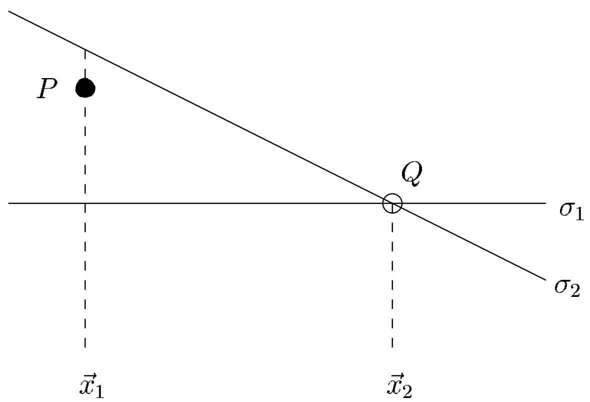

Figure 1. Illustration of the fact that the wavefunction is, in general, not a function on the spacetime continuum.

which is given by a superposition of two wavepackets $\chi^{(1)}$ and $\chi^{(2)}$ localized at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively,

$$\psi(x) = \chi^{(1)}(x) + \chi^{(2)}(x). \tag{1}$$

In the following we shall neglect, for simplicity, the extension as well as the spreading of these wavepackets. At the spacetime point P a position measurement is performed with the result that the electron is at P. Given such a situation we may consider two spacelike hypersurfaces σ_1 and σ_2 which intersect at the spacetime point Q (see figure 1). Both hypersurfaces appear as equal-time hypersurfaces in appropriately chosen coordinate frames K_1 and K_2, that is, there are observers O_1 and O_2 at rest in K_1 and K_2, respectively, such that σ_1 is an equal-time hypersurface for O_1, and σ_2 is an equal-time hypersurface for O_2. The important difference between both observers is that for O_2 the measurement has already taken place, whereas for O_1 it has not. Consequently, both observers assign different amplitudes to one and the same objective spacetime point Q, that is, $\psi(Q)$ on σ_1 is different from $\psi(Q)$ on σ_2. The conclusion is that ψ is no longer a function of the spacetime coordinates: The value of ψ at a spacetime point depends, in general, on the hypersurface crossing this point and to which it is associated. Thus, the wavefunction becomes a functional on the set of spacelike hypersurfaces and we may write

$$\psi = \psi(\sigma, x) \tag{2}$$

where x runs over the points of the hypersurface σ. We then have, for example, at the spacetime point Q of figure 1:

$$\psi(\sigma_1, Q) = \chi^{(2)}(Q) \neq \psi(\sigma_2, Q) = 0. \tag{3}$$

We have used here the prescription that the state vector reduction occurs along all spacelike hypersurfaces that cross the spacetime point P which gives the classical measuring event 'the electron is at P'. In fact, this is precisely the covariant state vector reduction postulate of Aharonov and Albert. There are other proposals for the state vector reduction [9] which have been demonstrated, however, to be in contradiction to the Hilbert space structure of quantum mechanics [8].

As remarked earlier the above results served as a starting point for our construction of a relativistic quantum state diffusion model. To this end, we use a covariant one-particle wave equation, the Dirac equation, and rewrite it as an equivalent evolution equation for the states $\psi(\sigma)$ attached to the various hypersurfaces σ,

$$\mathrm{d}\psi(\sigma) = -\mathrm{i}\mathrm{d}G(\sigma)\psi(\sigma). \tag{4}$$

The explicit form of the generator $\mathrm{d}G(\sigma)$ in this equation can be found in [1–3]. The important point to note here is that the derivation of this equation, which is presented in [1], is similar to the one used in the well-known Schwinger–Tomonaga formulation of relativistic quantum field theory. In fact, equation (4) can be obtained by projecting the Schwinger–Tomonaga equation of QED onto the one-electron sector.

Let us now look at equation (7) of Diósi's note. If we take the point x of that equation to be the point Q of our figure 1, then we find that, in view of our equation (3), the innocent looking equation (7) of Diósi is, in general, wrong! We shall give an example demonstrating this below. The question therefore arises: Does this imply a violation of covariance? The answer is, of course, no. To see this let us recall what is really meant by covariance [10]. Consider a Lorentz transformation

$$x \mapsto x' = \Lambda x + y. \tag{5}$$

Covariance means that there is a unitary representation $U(\Lambda, y)$ which relates the states in different frames K and K' through

$$\psi'(\sigma', x') = \left[U(\Lambda, y)\psi\right](\sigma', x') = S(\Lambda)\psi(\sigma, x) \tag{6}$$

such that the dynamical equation is invariant in form under these transformations.

This has been shown and discussed in detail in [1–3]. The important point to note here is that the transformation (6) is unitary with respect to the scalar product

$$\langle\psi(\sigma)|\phi(\sigma)\rangle_\sigma \equiv \int \frac{\mathrm{d}^3x}{n^0}\bar{\psi}(\sigma, x)n^\mu\gamma_\mu\phi(\sigma, x) \tag{7}$$

for the states on the flat, spacelike hypersurface σ, where n^μ denotes its unit normal vector. It is this expression by means of which probabilities and expectation values are determined in our approach. Taking now some observable $A(\sigma)$ on the hypersurface σ, we can express the transformation law for its expectation value as

$$\begin{aligned}\langle\psi(\sigma)|A(\sigma)|\psi(\sigma)\rangle_\sigma &\mapsto \langle\psi'(\sigma')|A(\sigma')|\psi'(\sigma')\rangle_{\sigma'}\\ &= \langle\psi(\sigma)|U^{-1}(\Lambda, y)A(\sigma')U(\Lambda, y)|\psi(\sigma)\rangle_\sigma.\end{aligned} \tag{8}$$

If $A(\sigma)$ is a scalar observable $A(\sigma, Q)$ which is strictly localized at some point Q on σ, we obtain for a Lorentz transformation which leaves invariant this point Q,

$$\langle\psi'(\sigma')|A(\sigma', Q)|\psi'(\sigma')\rangle_{\sigma'} = \langle\psi(\sigma)|A(\sigma, Q)|\psi(\sigma)\rangle_\sigma. \tag{9}$$

This is the correct transformation law for the situation discussed by Diósi. It looks quite similar to Diósi's equation (7). The decisive difference is, however, that in our equation (9) left- and right-hand side refer to the expectation values taken over *one and the same* hypersurface: σ and σ' represent *the same objective* hypersurface as seen from different inertial frames.

A completely different matter is to look at two *objectively different* hypersurfaces σ_1 and σ_2 as, for example, in figure 1. As we have remarked already Diósi's equation (7) is wrong at the point Q, where these surfaces intersect each other. To give an example we take the scalar observable $A(\sigma, Q)$ to be the operator

$$A(\sigma, Q) = \delta^{(3)}(\boldsymbol{x} - \boldsymbol{Q})n^0 \tag{10}$$

where n^0 is the time component of the 4-vector n, which is the unit normal vector of σ. By making use of (3) and (7) we then find for the expectation value along σ_2,

$$\langle\psi(\sigma_2)|A(\sigma_2, Q)|\psi(\sigma_2)\rangle_{\sigma_2} = 0 \tag{11}$$

whereas the expectation value along σ_1 yields

$$\langle\psi(\sigma_1)|A(\sigma_1, Q)|\psi(\sigma_1)\rangle_{\sigma_1} = \bar{\chi}^{(2)}(Q)n_1^\mu\gamma_\mu\chi^{(2)}(Q) \neq 0 \tag{12}$$

where n_1^μ is the unit normal vector of σ_1. This clearly demonstrates that Diósi's equation (7) is incorrect. Note further that the quantity on the right-hand side of equation (12) can be written as

$$\bar{\chi}^{(2)}(Q)n_1^\mu\gamma_\mu\chi^{(2)}(Q) = n_1^\mu j_\mu(Q) \tag{13}$$

where j_μ is the Dirac current. Thus we see that the expectation value (12) transforms as a scalar under Lorentz transformations which illustrates, for the present example, that our equation (9) is indeed correct, in contrast to Diósi's equation (7).

Although there is a Lorentz transformation that maps σ_1 onto σ_2, the states $\psi(\sigma_1)$ and $\psi(\sigma_2)$ are not, in general, unitarily equivalent. What covariance does require, however, is the invariance of the probabilities for *objective classical events*, as, for example, for the event to find the particle at P. By this we mean that the different observers passing the point P with different 4-velocities n, that is, with different equal-time hypersurfaces $\sigma(n)$ calculate the same probability prob(n, P) to find the electron at P. This is precisely what our theory predicts! To see this, we denote by $\pi(n)$ the projector which acts on the state space associated with the hypersurface $\sigma(n)$ through P, such that $\pi(n) = 1$ corresponds to the result that the electron is at P, whereas $\pi(n) = 0$ corresponds to the result that the electron in not at P. According to our theory $\pi(n)$ obeys the integrability condition [1, 2]

$$(\delta_\mu^\nu - n_\mu n^\nu)\frac{\partial\pi(n)}{\partial n^\nu} = -\mathrm{i}[W_\mu H(n), \pi(n)] \tag{14}$$

which ensures that $\psi(\sigma)$ is really a functional on the set of hypersurfaces. The observer passing the point P with equal-time hypersurface $\sigma(n)$ then obtains for the probability to find the electron at P:

$$\mathrm{prob}\,(n, P) = \langle\psi(\sigma(n))|\pi(n)|\psi(\sigma(n))\rangle_{\sigma(n)}. \tag{15}$$

Differentiating this expression with respect to n we get

$$\begin{aligned}(\delta_\mu^\nu - n_\mu n^\nu)\frac{\partial}{\partial n^\nu}\mathrm{prob}\,(n, P) &= \langle\psi(\sigma(n))|(\delta_\mu^\nu - n_\mu n^\nu)\frac{\partial\pi}{\partial n^\nu} + \mathrm{i}[W_\mu H, \pi]|\psi(\sigma(n))\rangle_{\sigma(n)} \\ &= 0\end{aligned}$$

where we have employed equation (35) of [1]. This equation means that

$$\mathrm{prob}(n, P) = \mathrm{prob}(P) \tag{16}$$

is independent of n, which expresses the relativistic invariance of the probability in our theory.

3. Disproving Diósi's counterexample

We now turn to a discussion of Diósi's 'counterexample'. We shall analyse this example fully relativistically and on the level of the stochastic wavefunction dynamics.

To begin with, we recall the parametrization of the flat, spacelike hypersurfaces which is used in our theory. Each hypersurface σ is uniquely specified by its unit normal vector n

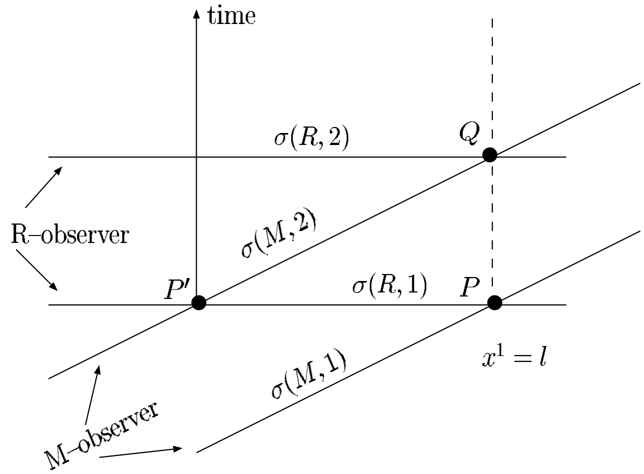

Figure 2. Sketch of the situation considered by Diósi.

and by its Lorentz distance a to some fixed point b. This means that the points x belonging to a hypersurface $\sigma = \sigma(n, a)$ are given by the equation

$$n(x - b) = a. \tag{17}$$

To see the physical meaning of this parametrization let O be some inertial observer moving through the spacetime point b. Denoting by τ the invariant proper time of O we can write the worldline $z(\tau)$ of O as $z(\tau) = n\tau + b$, where n is the constant 4-velocity of O. According to this equation the proper time of O is chosen such that for $\tau = 0$ he passes the spacetime point b. At a fixed proper time τ the hypersurface which appears as the equal-time hypersurface in O's rest frame is given by the equation $n(x - z(\tau)) = 0$. Inserting the expression for $z(\tau)$ we get $n(x - b) = \tau$. Comparing this relation with our definition (17) we see that the equal-time hypersurface in the rest frame of O at his proper time τ is given by $\sigma(n, a = \tau)$. Thus, the quantity a can be interpreted as the proper time of the observer O, n as his 4-velocity and b as the spacetime point through which O passes at $\tau = a = 0$. When discussing the covariance of our stochastic Dirac equation it is important to realize that these parameters transform as $n' = \Lambda n$, $a' = a$, $b' = \Lambda b + y$ under Lorentz transformations (5). In particular, the quantity a, being a proper time, transforms as a scalar and n, which is a unit normal vector, transforms as a tangential vector. Consequently, the quantity $s = an$ transforms as a tangential vector. This is the mistake in Diósi's argument put forward in his footnote [3]: he arrives at the erroneous conclusion that our theory breaks translational covariance† simply because he uses the wrong transformation rule $s' = \Lambda s + y$ for s, although the correct rule $s' = \Lambda s$ has been clearly stated in our papers.

Let us now turn to Diósi's 'counterexample'. The situation discussed by Diósi is depicted in figure 2. It is assumed that an electron has been prepared initially in a definite spin state described by a wavepacket which is localized at the spacetime point P with

† It seems that there is another misunderstanding in Diósi's comment. He always talks about *relativistic invariance* whereas we definitely mean *relativistic covariance*. We have never claimed that our theory is invariant. How can it be invariant? It describes an open system!

coordinates $x = (0, l, 0, 0)$ in the frame indicated in figure 2. We write this initial state as

$$\psi(n = (1, 0, 0, 0), a = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \varphi(\boldsymbol{x}) \tag{18}$$

which corresponds to the initial density matrix given in equation (10) of Diósi's comment. $\varphi(\boldsymbol{x})$ represents a normalized c-number function which is localized around the point P (given, for example, by a Gaussian of width larger than the Compton wavelength [10]). As before we shall neglect the extension of the wavepacket as well as its spreading. This means that we may neglect the Hamiltonian part in comparison to the stochastic part in the equation which describes the a-dependence of the wave function. Our stochastic Dirac equation [1] can therefore be written as (choosing units such that $\hbar = c = 1$)

$$\mathrm{d}_a\psi = -\tfrac{1}{2}(L(n, a) - \langle L(n, a)\rangle)^2\psi(n, a)\,\mathrm{d}a + (L(n, a) - \langle L(n, a)\rangle)\psi(n, a)\,\mathrm{d}W(a) \tag{19}$$

$$(\delta^\nu_\mu - n_\mu n^\nu)\frac{\partial\psi(n, a)}{\partial n^\nu} = -\mathrm{i}W_\mu H(n, a)\psi(n, a) \tag{20}$$

where $\mathrm{d}_a\psi \equiv \psi(n, a + \mathrm{d}a) - \psi(n, a)$, and $W_\mu = an_\mu - [x_\mu - b_\mu]$, $H(n, a) = \not{n}\gamma^0 H_D$, and H_D is the Dirac Hamiltonian which, for the present case, describes a free particle. Furthermore, we use the abbreviation $\langle L(n, a)\rangle \equiv \langle\psi(n, a)|L(n, a)|\psi(n, a)\rangle_{n,a}$.

Diósi considers two observers. One observer, called the R-observer, is at rest in the frame sketched in figure 2. Two of his equal-time hypersurfaces are given by $\sigma(R, 1)$ and $\sigma(R, 2)$. The corresponding unit normal vector of these surfaces is given by

$$n(0) = (1, 0, 0, 0). \tag{21}$$

The initial state given in equation (18) is thus a state on the hypersurface $\sigma(R, 1)$ of the R-observer. The other observer, called the M-observer, is moving with respect to the first one with velocity $\boldsymbol{v}$. His equal-time hypersurfaces have the unit normal vector

$$n(\boldsymbol{v}) = (\gamma, \gamma\boldsymbol{v}) \qquad \text{where} \qquad \gamma \equiv (1 - \boldsymbol{v}^2)^{-1/2}. \tag{22}$$

In figure 2 we have depicted two of the equal-time hypersurfaces of the M-observer, namely $\sigma(M, 1)$ and $\sigma(M, 2)$.

In his 'counterexample' Diósi compares the expectation values of an observable evaluated along the hypersurfaces $\sigma(R, 2)$ and $\sigma(M, 2)$ which intersect at the point Q. When determining these expectation values, he obviously uses the point P' as the base point of the parametrization (17), that is, the Lorentz distance a is measured from the point $b = P'$. What is wrong with this choice is that the equal-time hypersurfaces of the M-observer have passed the position of the initially localized electron at an earlier (proper) time. More precisely, there is an equal-time hypersurface $\sigma(M, 1)$ of the M-observer which intersects the point P and which lies totally in the past of $\sigma(M, 2)$. A comparison between the measurements of the R-observer and the M-observer, performed after the irreversible dynamical state vector localization, makes sense only if these observers agree on the initial preparation event and the corresponding initial state vectors. By some classical preparation event, the R-observer finds that the electron is at P and assigns the state $\psi(n(0), 0)$ to his equal-time hypersurface $\sigma(R, 1)$. In agreement with the R-observer, the M-observer therefore assigns a state to his equal-time hypersurface $\sigma(M, 1)$ which must be unitarily equivalent to $\psi(n(0), 0)$ and is thus connected to this state through equation (4). This implies that we have to choose $b = P$ in our parametrization (17). This follows from the fact that according to equation (20) the state $\psi(n(\boldsymbol{v}), 0)$ on the hypersurface $\sigma(M, 1)$ of the

M-observer is related to the state $\psi(n(0), 0)$ on the hypersurface $\sigma(R, 1)$ of the R-observer by the unitary dynamics of the Dirac equation.

Having fixed correctly the initial conditions for the stochastic dynamics we can now immediately disprove Diósi's 'counterexample'. The Lindblad operator in the rest frame of the R-observer is taken to be

$$L(n(0), a) = \frac{\sqrt{\Gamma}}{2}(1+\sigma_z)\tfrac{1}{2}(1+\gamma^0) \tag{23}$$

in accordance with Diósi's equation (11). By solving the integrability condition [1, 2] with the boundary condition (23) we get $L(n, a)$. For $a \gg 1/\Gamma$ the stochastic dynamics drives the initial states $\psi(n, 0)$ to one of two states $\psi^{\pm}(n, a)$ with probability $\frac{1}{2}$, where

$$\psi^{+}(n(0), a) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \varphi(\boldsymbol{x}) \qquad \psi^{-}(n(0), a) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \varphi(\boldsymbol{x}). \tag{24}$$

The meaning of these equations is the following. In the rest frame of the R-observer the state $\psi(n(0), 0)$ is driven to one of the states $\psi^{\pm}(n(0), a)$ which are given by the above expressions. In the M-observer's rest frame the same happens: his initial state $\psi(n(\boldsymbol{v}), 0)$ is driven to one of the states $\psi^{\pm}(n(\boldsymbol{v}), a)$. The important point is that these latter states are related to the states $\psi^{\pm}(n(0), a)$ of the R-observer through the unitary evolution equation (20). It follows that for a particular realization of the process the states $\psi(n, a)$ on *all* hypersurfaces with $a \gg 1/\Gamma$ are connected by solving the purely unitary evolution equation (4). Note that this is a direct consequence of the integrability condition [3], a fact, which is obviously missed by Diósi.

It should be clear that equation (4), being equivalent to the Dirac equation, does not depend in any way on the parameters n, a, and b by which we specify the hypersurfaces. Thus, we may take the point Q as the new base point of our parametrization, that is, we define the hypersurfaces $\sigma(n)$ which intersect the spacetime point Q by the equation

$$n(x - Q) = 0. \tag{25}$$

Consider two such hypersurfaces, namely $\sigma(R, 2)$ and $\sigma(M, 2)$, which intersect the point Q, as is indicated in figure 2. We measure the observable $A(n)$ along these hypersurfaces. $A(n)$ is determined as follows. On $\sigma(R, 2) = \sigma(n(0))$ it is given by σ_x in agreement with equation (8) of Diósi. It follows with the help of equation (24) that

$$\langle\psi^{\pm}(n(0))|A(n(0))|\psi^{\pm}(n(0))\rangle_{n(0)} = 0. \tag{26}$$

The observable $A(n(\boldsymbol{v}))$ on the equal-time hypersurface $\sigma(M, 2) = \sigma(n(\boldsymbol{v}))$ of the M-observer is determined by solving the integrability condition

$$(\delta_\mu^\nu - n_\mu n^\nu)\frac{\partial A(n)}{\partial n^\nu} = -\mathrm{i}[W_\mu H(n), A(n)]. \tag{27}$$

In the same way as in section 2 we find

$$(\delta_\mu^\nu - n_\mu n^\nu)\frac{\partial}{\partial n^\nu}\langle\psi^{\pm}(n)|A(n)|\psi^{\pm}(n)\rangle_n = 0 \tag{28}$$

which means that the expectation value $\langle\psi^{\pm}(n)|A(n)|\psi^{\pm}(n)\rangle_n$ is independent of n. Hence we have

$$\langle\psi^{\pm}(n(0))|A(n(0))|\psi^{\pm}(n(0))\rangle_{n(0)} = \langle\psi^{\pm}(n(\boldsymbol{v}))|A(n(\boldsymbol{v}))|\psi^{\pm}(n(\boldsymbol{v}))\rangle_{n(\boldsymbol{v})} = 0. \tag{29}$$

This equation shows that both observers predict the same expectation value, namely 0. Diósi's equation (14) is therefore wrong and there are no inconsistencies in our theory. Thus, Diósi's 'counterexample' has been disproven.

We conclude with two remarks. First, we have discussed the example on the level of the stochastic wavefunction dynamics. It should be clear that our results are also valid on the level of the density matrix which is obtained by taking the ensemble average. Second, since our arguments are valid for the full relativistic theory, they hold to all orders in v/c and also in the non-relativistic limit, of course. It is demonstrated in [3] that our relativistic formulation has indeed a well-defined non-relativistic limit which leads to the Ito equation of the quantum state diffusion model [4].

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